Riemann's quadratic relations of Selberg-type integrals

Koji Cho

Kyushu U., Japan

Abstract

Let $\ell_i = \ell_i(t)$ be an affine linear form in $t = (t_1, \dots, t_n)$, and a_i a real non-integral constant, for $1 \le i \le r$. Put

$$L_i := \{t \in \mathbf{C}^n \mid \ell_i(t) = 0\}, \quad X := \mathbf{C}^n - \cup_i L_i,$$

and consider the (multi-valued) function $u := \prod_{i=1}^r \ell_i(t)^{a_i}$ on X, which defines a local system $\mathcal{L} = \mathcal{L}_u$. Under some generic condition on L_i and a_i, the pair $(X, \mathcal{L})$ is purely n-(co)dimensional, i.e.

$$H_j(X, \mathcal{L}) = 0, \quad H^j(X, \mathcal{L}) = 0, \quad \text{if } j \ne n.$$

We are interested in evaluating the integral

$$\int_{\mathbf{C}^n} |u|^2 \, dt \wedge d\bar{t}, \quad \text{where} \quad dt = dt_1 \wedge \cdots \wedge dt_n$$

in terms of the periods

$$\int_{\gamma_k} u \, dt,$$

where $\{\gamma_k\}$ form a basis of $H_n(X, \mathcal{L})$, and the intersection numbers $\gamma_k \bullet \check{\gamma}_l$, where $\check{\gamma}_k$ is an element of $H_n(X, \check{\mathcal{L}})$ corresponding to $\gamma_k \in H_n(X, \mathcal{L})$. The simplest non-trivial example is

$$\int_{\mathbf{C}} |t|^{2\alpha} |1-t|^{2\beta} \, dt \wedge d\bar{t}.$$

This is well-known to be equal to

$$B(\alpha+1, \beta+1)^2 \, \frac{(1-e^{2\pi i\alpha})(1-e^{2\pi i\beta})}{1-e^{2\pi i(\alpha+\beta)}},$$

where B is the Beta function

$$B(\alpha+1, \beta+1) = \int_0^1 t^\alpha (1-t)^\beta dt.$$

The factor $(1-e^{2\pi i\alpha})(1-e^{2\pi i\beta})/(1-e^{2\pi i(\alpha+\beta)})$ is the reciprocal of the intersection number of the cycles $(0,1) \otimes t^\alpha(1-t)^\beta$ and $(0,1) \otimes t^{-\alpha}(1-t)^{-\beta}$.

In this talk, we give a similar formula for the integral

$$\int_{\mathbf{C}} \prod_{i=1}^r |t - x_i|^{2\alpha_i} \, dt \wedge d\bar{t}.$$

We also consider integrals of Selberg type:

$$\int_{\mathbf{C}^n} \prod_{i=1}^{n} |t_i|^{2\alpha_i} |1-t_i|^{2\beta_i} \prod_{1\le i<j\le n} |t_i - t_j|^{2g_{ij}} \, dt \wedge d\bar{t},$$

and

$$\int_{\mathbf{C}^n} \prod_{i=1}^{n} |t_i|^{2\alpha_i} |1-t_i|^{2\beta} |z_i - t_i|^{2\gamma_i} \prod_{1\le i<j\le n} |t_i - t_j|^{2g_{ij}} \, dt \wedge d\bar{t}.$$